Quantum Morphisms

Lecture 4

Last Week

1) If G has a quantum n-coloring, it has one using projections of all the same rank (χ_q^r + rank-r colorings).

2) Rank-r quantum n-coloring E_{gi} gives $g \mapsto E_{g_1}$ (nr/r)-projective representation.

3) $\xi_f(G) = \inf\{\frac{d}{r} : G$ has a (d/r)-representation$\} \leq \chi_q(G), \chi_f(G), \xi(G)$

4) $\xi(G) = \min\{d : G$ has an orthogonal representation in $\mathbb{C}^d\} \leq \chi_q^1(G)$

5) Flat ortho. rep. of G in $\mathbb{C}^d \Rightarrow \chi_q^1(G) \leq d$

6) Real ortho. rep. in dimension 4 (or 8) $\Rightarrow \chi_q^1(G) \leq 4$ (or 8)

7) To construct examples with $\chi_q(G) < \chi(G)$: Consider orthogonality graphs of appropriate sets of vectors and hope.

Perhaps $\xi(G) \overset{\leq}{=} \chi_q^1(G) \overset{\geq}{=} \chi_q(G)$?

Fukawa, Imai, + Le Gall: Use reduction 3-SAT → 3-COLORING to obtain graph with $\chi_q(G) = 3 < \chi(G)$.

This implies that $\chi_q(G) < \chi_q^1(G)$ (HW: $\chi_q^1(G) = 3 \Rightarrow \chi(G) = 3$)

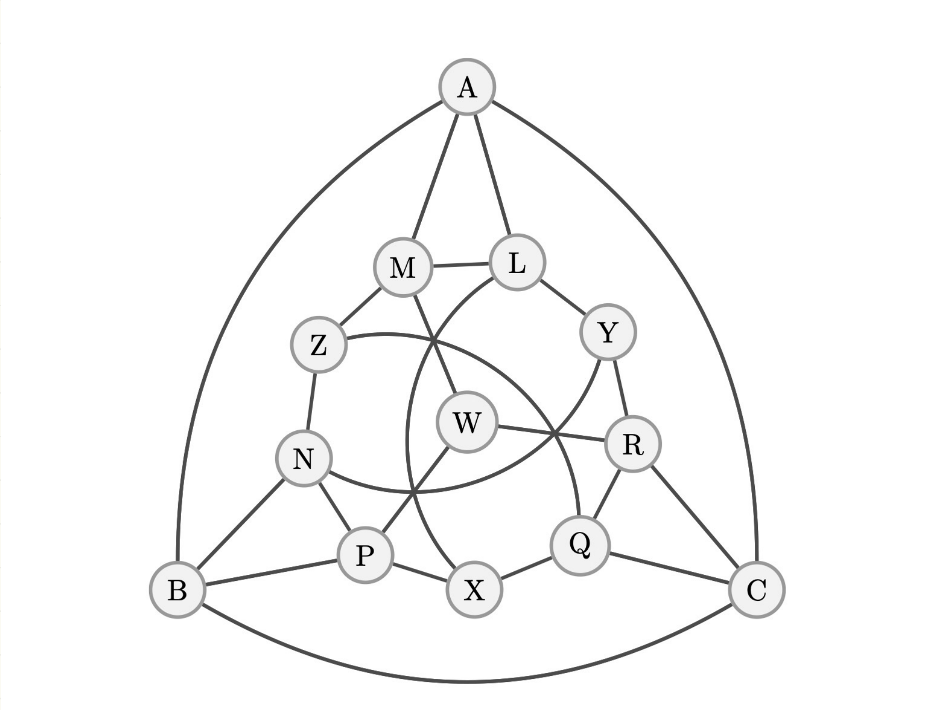
Mančinska + me: $\chi_q(G) < \xi(G)$ also possible. Oddities of Quantum Coloring

Thus $\xi(G) \leq \chi_q(G)$ does not hold in general.

What about $\chi_q(G) \leq \xi(G)$ for all graphs G?

G_{13} - orthogonality graph of the vectors in $\mathbb{R}^3$ with entries from $\{0, 1, -1\}$.

$\omega(G_{13}) = \xi(G_{13}) = 3 < 4 = \chi(G_{13})$



$\chi_q(G_{13}) = ?$ $4 > 3 = \xi(G_{13})$

What about $G_{14} := G_{13}$ + apex vertex?

$\chi(G_{13}) = 4 \Rightarrow \chi(G_{14}) = 5$

G_{13} has ortho. rep. in $\mathbb{R}^3 \Rightarrow G_{14}$ has ortho. rep. in $\mathbb{R}^4 \Rightarrow \chi_q(G_{14}) \leq 4$.

Remarks: 1) $\chi_q(G_{14}) < \chi(G_{14})$ Smallest known example.

2) $\chi_q(G_{13}) = \chi_q(G_{14})$ Adding an apex vertex did not increase χ_q!!!

Exercise: Show that if G + apex vtx has a locally commuting quantum n-coloring, then G has a locally commuting quantum $(n-1)$-coloring, i.e.

$G \to H$

$E_{gh} E_{g'h'} = E_{g'h'} E_{gh}$ if $g \sim g'$

$$\chi_q^{l.c.}(G + \text{apex}) = \chi_q^{l.c.}(G) + 1$$

The locally commuting constraint arises in a slightly different (G,H)-homomorphism game introduced by Abramsky, Barbosa, de Silva, & Zapata

Projective Packings

An assignment $g \mapsto E_g \in \mathbb{C}^{d \times d}$ of projections to the vertices of a graph G is a projective packing if $g \sim g' \Rightarrow E_g E_{g'} = 0$.
The value of a projective packing is $\frac{1}{d} \sum_{g \in V(G)} rk(E_g)$.

Projective packing number:

$$\alpha_p(G) := \sup\{ x \in \mathbb{Q} : G \text{ has a proj. pack. of value } x \}.$$

$$\omega_p(G) := \alpha_p(\bar{G}) \quad \text{(projective clique number)}$$

Properties

1) $S \subseteq V(G)$ indpt set, $E_g = \begin{cases} I & g \in S \\ 0 & \text{o.w.} \end{cases}$ is a proj. pack. of value $|S|$.
Thus $\alpha(G) \leq \alpha_p(G)$.

2) $\xi_f(G) \geq \frac{|V(G)|}{\alpha_p(G)}$ with equality if G is vertex transitive.

Proof: Suppose $g \mapsto E_g$ is a (d/r)-representation.
Then it is also a proj. pack. of value

$$\frac{1}{d} \sum_{g \in V(G)} rk(E_g) = \frac{r|V(G)|}{d}.$$

Thus $\alpha_p(G) \geq \frac{r|V(G)|}{d} = \frac{|V(G)|}{(d/r)} \Rightarrow \frac{d}{r} \geq \frac{|V(G)|}{\alpha_p(G)}$.

3) $\omega_p(G) \leq \xi_f(G)$ $\omega(G) \leq \chi_f(G)$

<u>Proof</u>: $g \mapsto E_g \in \mathbb{C}^{k \times k}$ proj. clique of value $\frac{R}{k} = \frac{1}{k} \sum_g rk(E_g)$.

$g \mapsto F_g \in \mathbb{C}^{d \times d}$ a (d/r)-representation.

$P_g := E_g \otimes F_g \in \mathbb{C}^{kd \times kd}$.

Then P_g is a projection for all $g \in V(G)$ + $P_g P_{g'} = 0$ if $g \neq g'$.

Thus $rk(\sum_g P_g) = \sum_g rk(P_g) = \sum_g rk(E_g \otimes F_g) = r \sum_g rk(E_g) = rR$.

But, $rk(\sum_g P_g) \leq kd$. Therefore, $rR \leq kd \Rightarrow \frac{R}{k} \leq \frac{d}{r}$.

4) If $G \xrightarrow{q} H$ then $\omega_p(G) \leq \omega_p(H)$ + $\xi_f(G) \leq \xi_f(H)$.

<u>Proof</u>: $g \mapsto E_g \in \mathbb{C}^{k \times k}$ proj. clique of value $\frac{R}{k} = \frac{1}{k} \sum_g rk(E_g)$.

$F_{gh} \in \mathbb{C}^{d \times d}$ for $g \in V(G), h \in V(H)$ give quantum hom from G to H.

$P_h := \sum_g E_g \otimes F_{gh}$

P_h is a projection, $P_h P_{h'} = 0$ if $h \sim h'$ in $\overline{H}$

value of $P_h \geq$ value of E_g

An analogy: α_p is to ξ_f as α is to χ_f.

Shorthand: $\alpha_p(G) \dot{\geq} x \in \mathbb{Q}$ means G has a proj. pack. of value (at least) x.

$\alpha_p(G) \doteq x \in \mathbb{Q}$ means $\alpha_p(G) = x$ & $\alpha_p(G) \dot{\geq} x$.

Quantum Independence Number

$\omega_q(G) = \max\{n : K_n \xrightarrow{q} G\}$

$\alpha_q(G) = \omega_q(\overline{G})$

Projections $E_{ig} \in \mathbb{C}^{d \times d}$ give a quantum hom. $K_n \xrightarrow{q} \overline{G}$ if

1) $\sum_g E_{ig} = I \quad \forall\, i \in [n]$

2) $E_{ig} E_{jg'} = 0$ if $[i \neq j$ & $(g \sim g'$ or $g = g')]$ or $[i = j$ & $g \neq g']$

$F_g := \sum_{i=1}^{n} E_{ig}$ is a projective packing of value n.

Therefore, $\alpha_p(G) \dot{\geq} \alpha_q(G)$.

Proof: Exercise.

Open Problem:

$\alpha_q(G) = \min\{\lfloor x \rfloor : G$ has proj. pack. of value $x\}$?

$\neq \lfloor \alpha_p(G) \rfloor$

A Construction

Suppose $g \mapsto |\psi_g\rangle \in \mathbb{C}^d$ is an ortho. rep. of G, and $f: V(G) \to [k]$ is a k-coloring of $\overline{G}$ where $k = \frac{|V(G)|}{d}$.

Then $\omega(G) = d$ & $f^{-1}(i)$ is a clique of size d $\forall i \in [k]$.

Therefore, $\{|\psi_g\rangle : g \in f^{-1}(i)\}$ is an orthonormal basis $\forall i$.

Define $E_{ig} = \begin{cases} |\psi_g\rangle\langle\psi_g| & \text{if } g \in f^{-1}(i) \\ 0 & \text{o.w.} \end{cases}$

Claim: The E_{ig} give a quantum hom $K_k \xrightarrow{q} \overline{G}$.

Proof: $\sum_{g \in V(G)} E_{ig} = \sum_{g \in f^{-1}(i)} |\psi_g\rangle\langle\psi_g| = I$.

$i \neq j$: $E_{ig} E_{jg} = 0$ because $f^{-1}(i) \cap f^{-1}(j) = \emptyset$.

$i \neq j$ & $g \sim g'$: $E_{ig} E_{jg'} = \begin{cases} |\psi_g\rangle\langle\psi_g|\psi_{g'}\rangle\langle\psi_{g'}| \overset{0}{=} & \text{if } g \in f^{-1}(i) \text{ \& } g' \in f^{-1}(j) \\ 0 & \text{o.w.} \end{cases}$

Theorem (Cubitt, Leung, Mattews, & Winter):

If $\chi(\overline{G}) = \frac{|V(G)|}{\xi(G)}$ then $\alpha_q(G) = \chi(\overline{G})$.

$\leq$ always true

Theorem (Mančinska, Scarpa, & Severini):

$\alpha_q(G) = \chi(\overline{G})$ if & only if $\alpha_p(G) \doteq \chi(\overline{G})$.

How do we find such graphs that also have $\alpha(G) < \chi(\overline{G})$?

Kochen-Specker set (KS set): A set $S \subseteq \mathbb{C}^d$ of nonzero vectors such that there is no subset $T \subseteq S$ of mutually non-orthogonal vectors containing one vector from every orthogonal basis $B \in \mathcal{B}$ – set of bases in S

$V(G(S)) = \{ (|\psi\rangle, B) : |\psi\rangle \in B \subseteq S,\ B \text{ orthogonal basis} \}$.

$(|\psi\rangle, B) \sim (|\psi'\rangle, B')$ if $\langle \psi | \psi' \rangle = 0$.

$n := |V(G(S))|$

$b :=$ # of orthogonal bases in S $\left(= \frac{n}{d} \right)$.

$(|\psi\rangle, B) \mapsto B$ is a coloring of $\overline{G(S)}$ with $b = \frac{n}{d}$ colors.

$(|\psi\rangle, B) \mapsto |\psi\rangle$ is an ortho. rep. of G in $\mathbb{C}^d$.

Thus $\alpha_q(G(S)) = \chi(\overline{G(S)}) = b = \frac{n}{d}$.

But an indpt set of size b would hit every orthogonal basis B exactly once, a contradiction.

Thus $\alpha(G(S)) < b = \alpha_q(G(S))$. $\Rightarrow \chi(G) \geq \frac{n}{\alpha} > \frac{n}{b} = d = \xi(G)$

A Kochen-Specker set

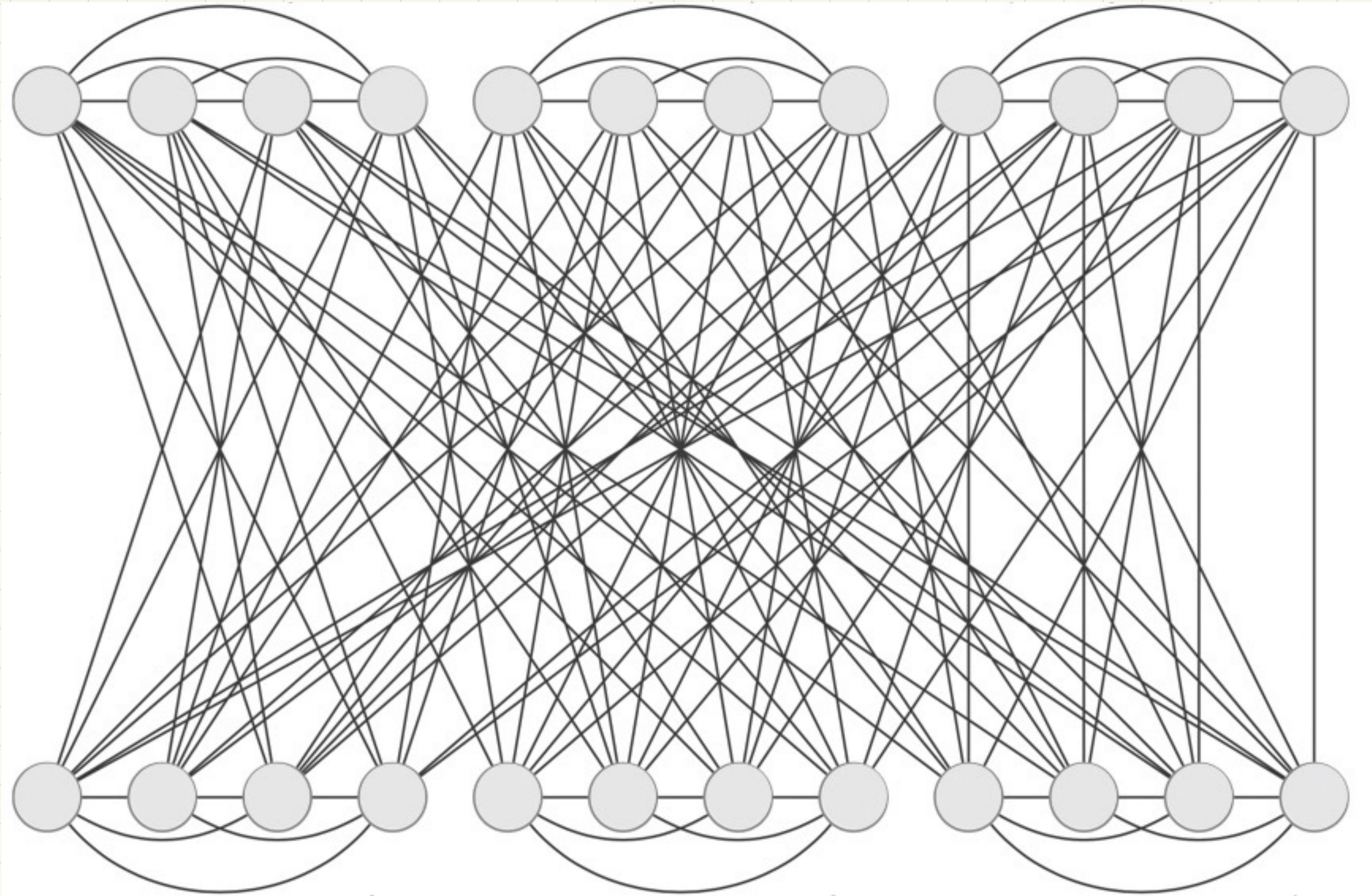
1 : $(1, 0, 0, 0)$	2 : $(0, 1, 0, 0)$	3 : $(0, 0, 1, 0)$	4 : $(0, 0, 0, 1)$
5 : $(0, 1, 1, 0)$	6 : $(1, 0, 0, -1)$	7 : $(1, 0, 0, 1)$	8 : $(0, 1, -1, 0)$
9 : $(1, 1, 1, 1)$	10 : $(1, -1, 1, -1)$	11 : $(1, -1, -1, 1)$	12 : $(1, 1, -1, -1)$
13 : $(1, -1, 0, 0)$	14 : $(1, 1, 0, 0)$	15 : $(0, 0, 1, 1)$	16 : $(0, 0, 1, -1)$
17 : $(-1, 1, 1, 1)$	18 : $(1, 1, 1, -1)$	19 : $(1, -1, 1, 1)$	20 : $(1, 1, -1, 1)$
21 : $(1, 0, 1, 0)$	22 : $(0, 1, 0, 1)$	23 : $(1, 0, -1, 0)$	24 : $(0, 1, 0, -1)$

$$\chi(\bar{G}) \geq \chi_q(\bar{G}) \geq \xi_f(\bar{G}) \geq \alpha_p(G) \geq \alpha_q(G) \geq \alpha(G)$$

$\chi_f(\bar{G})$

$\xi(G)$

The orthogonality graph



$\alpha_q(G) = 6 > 5 = \alpha(G)$ ALSO $\chi_q(G) = 4 < \frac{24}{5} \leq \chi(G) = 5$